Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka, Surathkal

Odd Semester (2014- 2015) **Examination: Mid Sem**
Course Code: MA200 **Course Name: Mathematical Foundations of Information Technology**
Date:08/09/2014 **Time:08.30 AM to 10.00 AM** **Maximum Marks: 50**

Q.1. (A) Prove that every graph on n vertices and k edges has at least $(n-k)$ components. [04]
Ans.: An n-vertex graph with no edges has n components. Adding an edge decreases the number of components by 0 or 1. Hence each edge added reduces the number of components by at most one. Therefore, when k edges have been added, the number of components is still at least $n-k$.

(B) Show that if n people attend a party and some shake hands with others (but not with themselves), then at the end, there are at least two people who have shaken hands with the same number of people. [06]
Ans.: This problem can be looked upon as a graph with n vertices where each person is a vertex and shaking hands is like an edge. Thus, it is enough to show that every graph has at least two vertices of the same degree. Degree of a vertex can vary between 0 and $n-1$; thus, we have n choices of degrees. But, if there is a vertex of degree zero, then the maximum degree can only be $n-2$; thus, we have a choice of only $n-1$ distinct values. So, by the Pigeonhole Principle, there must be at least two vertices of the same degree. On the other hand, if there is no vertex of degree zero, then also we have the choice only $n-1$ distinct values. Therefore, by a similar argument as above, the graph has at least two vertices of the same degree.

Q.2. (A) If $k > 0$, then prove that every k-regular bipartite graph has the same number of vertices in each partite set. [04]
Ans.: Let G be k-regular bipartite graph and A and B be the partition of the vertex set. Since G is regular, every vertex must be of the same degree. Since G is bipartite, every vertex in one partition must be adjacent to some vertices in the other partition only. Therefore, the total number of edges $e(G)$ is $k \cdot |A$ and so also, $k \cdot |B|$. Thus $k \cdot |A| = k \cdot |B|$; whence, $|A| = |B|$.

(B) Let G be a graph on n vertices, namely $v_1,\ v_2, \ldots.., \ v_n$, where $n \geq 3$. If $e(G)$ denotes the number of edges in G and $d_G(v_i)$ denotes the degree of a vertex v_i in G, then prove that [06]

(i) $e(G) = \frac{\sum_{i=1}^{n} e(G-v_i)}{n-2}$
Ans.: An edge $e \in G$ appears in $G - v_i$ if and only if v_i is not an end vertex of e. Thus, $\sum\limits_{i=1}^{n}$ counts each edge exactly $n-2$ times. Hence the result.

(ii) $d_G(v_i) = \frac{\sum_{i=1}^{n} e(G-v_i)}{n-2} - e(G - v_j)$
Ans.: The degree of v_j can be computed as the number of edges lost when deleting v_j to form $G - v_j$. Hence, $d(v_j) = e(G) - e(G - v_j)$ and the result follows.

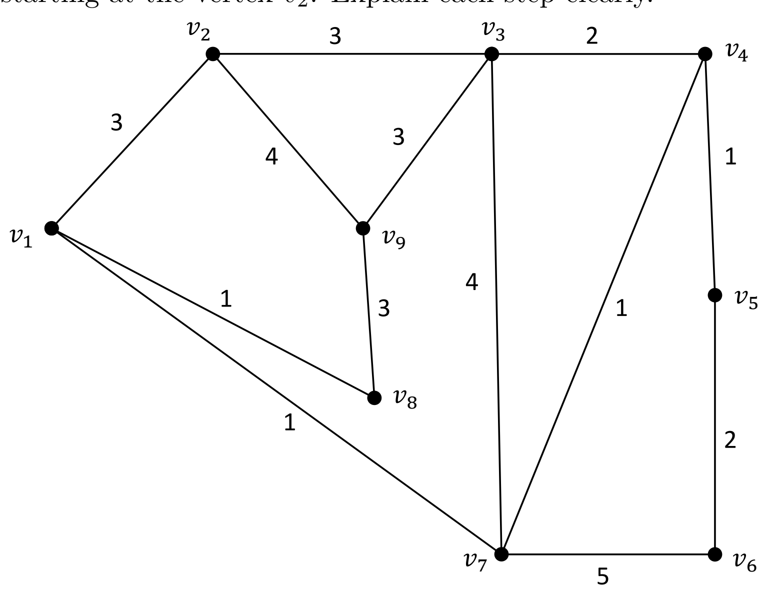
Q.3. (A) Prove that if G is a graph such that every vertex is of even degree, then the edges can be partitioned into cycles. [04]
Ans.: Without loss of generality, assume that G is connected. We use induction on the number of edges, say m; the result being trivial for $m = 0, 1, 2, 3$.
Assume that the result is true for every graph with fewer than m edges. Suppose G is such a graph on m edges. Then G contains a cycle, say C. Remove the edges on the cycle

C; the resulting graph, say H must still be a graph with all its vertices of even degree or degree zero. Therefore, by the induction hypothesis, the edges in H can be partitioned into cycles; thus, G is a graph which has a partition of its edges into these cycles and C.

(B) If G is an Eulerian graph then prove that $L(G)$ is both Eulerian and Hamiltonian. [06]
Ans.: Since G is Eulerian, every vertex of G must be of even degree. Now, any edge $x = uv$ of G is represented by a vertex whose degree is $deg\ u + deg\ v - 2$. Thus, the degree of every vertex in $L(G)$ also must be even and hence $L(G)$ must be Eulerian. Further, the Eulerian Circuit of G corresponds to a cycle in $L(G)$ and spans all the vertices of $L(G)$. Hence $L(G)$ is Hamiltonian, too.

Q.4. (A) If the degree of every vertex in a $(n,\ m)$-graph G (where $n \geq 2$) is at least $\frac{(n-1)}{2}$, then prove that G is a connected graph. [04]
Ans.: Clearly, $\delta(G) \geq \frac{n-1}{2}$. Then there is at least one component with $\frac{n-1}{2} + 1$ vertices, that is, $\frac{n+1}{2}$ vertices. That means, the remaining components have $n - \frac{n+1}{2} = \frac{n-1}{2}$ vertices, meaning that degrees of the vertices in the remaining components is not more than $\frac{n-1}{2} - 1$, which is a contradiction. Thus, there must be only one component and hence G must be connected.

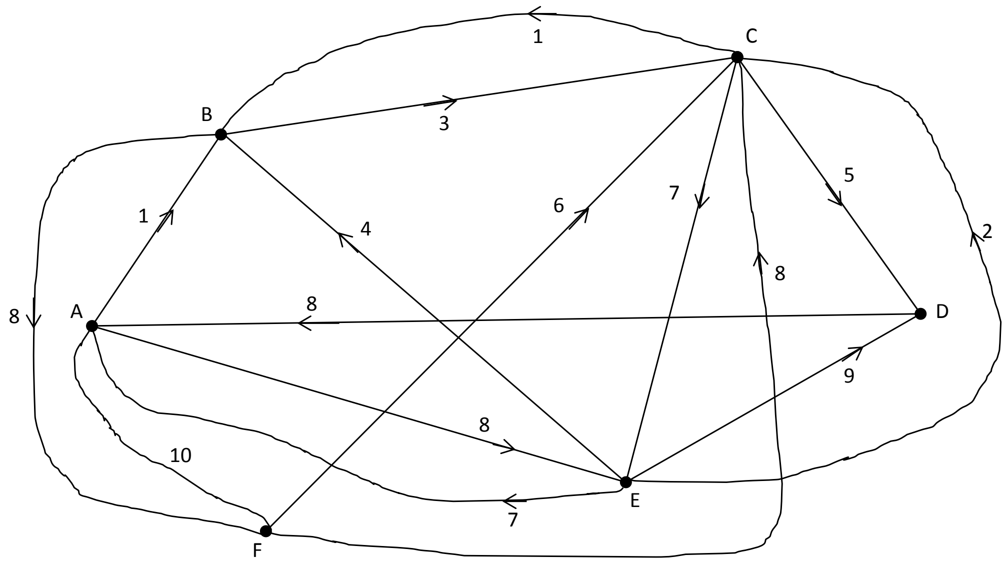
(B) Implement Prim's Algorithm to find a minimal spanning tree of the following network, starting at the vertex v_2. Explain each step clearly. [06]



Ans.: The implementation of Prim's Algorithm is a bookwork. I think, the total minimal weight is 14.

Q.5. (A) Show that a tree with n vertices has exactly $(n-1)$ edges. [04]
Ans.: We prove the result using induction on the number of vertices n, the result being true for $n = 1$.
Assume that the result is true for every tree with fewer than n vertices and let T be a tree on n vertices. Consider any edge e of T. $T - e$ is clearly a disconnected graph (since every edge of a tree is a bridge) with two components T_1 and T_2. By the induction hypothesis, each of T_1 and T_2 has one more vertex than the number of edges in it. Thus, in T, which includes e, there must be one more vertex than the number of edges in it.

(B) Implement Dijkstra's Algorithm to find the shortest path and the distance from the vertex A to all the remaining vertices of the following network. Each step must be written clearly. [06]



Ans.: The implementation of Dijkstra's Algorithm is a bookwork. For any instance of multiple edges, the algorithm logically chooses the lesser weight and hence eventually, we get the shortest path and the distances. The final answer is as follows:

A	B	C	D	E	F
bestd	1	4	9	8	9
tree	A	B	C	A	B

(*This document contains* 3 *page(s) and* 5 *Questions with the Model/ Outline of Answers.*)