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ON THE SPLITTING GRAPH OF A GRAPH

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For each point \( v \) of a graph \( G \), take a new point \( v' \). Join \( v' \) to all points of \( G \) adjacent to \( v \). The graph \( S(G) \) thus obtained is called the splitting graph of \( G \) (See Fig. 1).

![Fig. 1](image)

We study some properties of \( S(G) \) and obtain a characterization of splitting graphs.

We follow the notations and terminology given in [1].

Proposition 1: (i) If \( G \) is a \((p,q)\) graph, then \( S(G) \) is a \((2q,3q)\) graph,
(ii) For any point \( v \) in \( G \),
\[ \text{deg } v = 2 \text{ deg } v \text{ and } \text{deg } v' = \text{deg } v, \]
\[ S(G) \quad G \quad S(G) \quad G \]
where \( v' \) is the point as defined above.
(iii) If \( G \) has \( t \) triangles, then \( S(G) \) has \( 4t \) triangles.

**Proof:** (i), (ii) and (iii) follow directly from the definition of \( S(G) \). Also (iv) is true, since every triangle in \( G \) gives rise to four triangles in \( S(G) \).

**Proposition 2:**

\[ \psi(S(G)) = \psi(G) \]

**Proof:** For each point \( v \) in \( G \), let \( v' \) be the new point chosen in the construction of \( S(G) \). Any \( \psi(G) \) coloring of \( G \) can be extended to \( S(G) \), since we can assign the same colour to \( v \) and \( v' \).

As usual, let \( \alpha_0, \alpha_1, \beta_0 \) and \( \beta_1 \) respectively denote the point covering number, line covering number, point independence number and line independence number of a graph.

**Proposition 3:** For any graph \( G \) with \( p \) points,

(i) \( \alpha_0(S(G)) = p = \beta_0(S(G)) \).

(ii) \( \alpha_1(S(G)) = 2\alpha_1(G) \) and \( \beta_1(S(G)) = 2\beta_1(G) \).

**Proof:** Let \( V \) be the vertex set of \( G \) and \( V' \) be the set of new points introduced in the construction of \( S(G) \). We observe that \( V' \) is an independent set with \( p \) points and hence \( \beta_0(S(G)) \geq p \). Suppose \( B \) is an independent set with more than \( p \) points. Then \( B \) should contain points of both \( V \) and \( V' \). Let \( |V \cap B| = r \). Then these \( r \) points of \( V \cap B \) are adjacent to at least \( r \) points of \( V' \). So, there can be at most \( p-r \) points of \( V' \) in \( B \). Thus \( B \) can have at most \( p \) points. Thus \( \beta_0(S(G)) = p \). Since \( S(G) \) has \( 2p \) points and \( \alpha_0(S(G)) + \beta_0(S(G)) = 2p \), we have \( \alpha_0(S(G)) = p \). Now prove \( \beta_1(S(G)) = 2\beta_1(G) \). We observe that corresponding to each line \( uv \) of \( G \), we have three lines \( u'v, uv' \) and \( uv \) in \( S(G) \). The lines \( u'v \) and \( uv' \) are independent. Thus, each line of \( G \) gives rise to two independent lines in \( S(G) \). So \( \beta_1(G) \) independent lines of \( G \) give \( 2\beta_1(G) \) independent lines in \( S(G) \). It is not hard to see that this is the maximum number of independent lines in \( S(G) \).

Hence, \( \beta_1(S(G)) = 2\beta_1(G) \).

Now, from the equation,

\[ \beta_1(S(G)) + \alpha_1(S(G)) = 2p = 2\beta_1(G) + 2\alpha_1(G) \]

we have \( \alpha_1(S(G)) = 2\alpha_1(G) \).

**Proposition 4:** \( S(G) - E(G) = G \oplus K_2 \), where \( E(G) \) is the edge set of \( G \) and \( G \oplus K_2 \) is the tensor product of \( G \) with \( K_2 \).

**Proof:** We recall that the tensor product \( G_1 \oplus G_2 \) has vertex set \( V(G_1) \times V(G_2) \) where two vertices \((u,v)\) and \((w,z)\) are adjacent if and only if \( u \) adj \( w \) and \( v \) adj \( z \). If \( G \) has \( p \) vertices, then \( G \oplus K_2 \) has \( 2p \) vertices.
The product $G \oplus K_2$ can also be defined as follows: Let $V$ be the vertex set $G$. Let $V'$ be the set, such that

$V \cap V' = \emptyset$ and $|V| = |V'|$.

Let $v \rightarrow v'$ be a objective mapping of $V$ and $V'$. Now construct a graph $H$ with vertex set $V \cup V'$ whose edges are given as follows: $uv'$ and $vu'$ are edges in $H$, if and only if $uv$ is an edge in $G$. The graph $H$ thus obtained is $G \oplus K_2$.

We can consider the set $V'$ as the set of new points introduced to construct $S(G)$ from $G$, and for a point $v$ in $G$, let $v'$ denote the corresponding new point in $V'$. As observed earlier, each line $uv$ in $G$ gives three lines $uv'$, $u'v$, $uv$ in $S(G)$. We note that $uv'$ and $u'v$ are lines in $G \oplus K_2$. Thus, it follows that, if we remove lines of $G$ in $S(G)$, we get $G \oplus K_2$.

One can easily verify the following:

**Proposition 5:**

(a) $S(G)$ is planar, if

(i) $G$ is a tree

(ii) $G$ is an even cycle.

(b) $S(G)$ is non-planar if

(i) $G$ is an odd cycle of length $\geq 5$

(ii) $G$ is non-outer planar,

(iii) $G$ contains a cycle with a chord.

Characterization of graphs whose splitting graphs are planar, is an open problem.

We now obtain a characterization of splitting graphs.

**Theorem 1:** A graph $G$ is a splitting graph if and only if $V(G)$ can be partitioned into two sets $V_1 \cup V_2$ such that (i) there exists a bijective mapping $v_1 \rightarrow v_2$ from $V_1$ to $V_2$, and

(ii) $N(v_2) = N(v_1) \cap N_1$,

where $N(v) = \{u : uv \notin E(G)\}$.

**Proof:** Necessity: Let $G = S(H)$, for some graph $H$. To construct $G$ from $H$, we add a new point $v'$ for each point $v$ of $H$, and make it
adjacent to all points of \( H \) to which \( v \) is adjacent. Let \( V_1 = V(H) \), and 
\( V_2 = V(G) - V(H) \). For \( v_1 \in V_1 \), let \( v_2 \in V_2 \), be the new point introduced. 
Then \( v_1 \ v_2 \) defines a bijective mapping of \( V_1 \) onto \( V_2 \) and \( N(v_2) = N(v_1) \cap V_1 \).

**Sufficiency:** Let the given conditions be true for a graph \( G \). Let 
\( H \) be the subgraph of \( G \) induced by \( V_1 \). It is easy to see that \( G = S(H) \).

**REFERENCE**